Journal of Logic, Language and Information **xxx:** 1–31, 2004.

Abductive Logics in a Belief Revision Framework

BERNARD WALLISER
CERAS, Ecole Nationale des Ponts et Chaussées and Ecole des Hautes Etudes en Sciences Sociales, Paris, France
E-mail: walliser@mail.enpc.fr

DENIS ZWIRN
CREA, Ecole Polytechnique, Paris, France
E-mail: dzwirn@numilog.com

HERVÉ ZWIRN
CMLA, Ecole Normale Supérieure de Cachan and Institut d'Histoire et de Philosophie des Sciences et des Techniques, CNRS, France
E-mail: herve.zwirn@m4x.org

(Received 17 December 2003; in final form 14 January 2004)

Abstract. Abduction was first introduced in the epistemological context of scientific discovery. It was more recently analyzed in artificial intelligence, especially with respect to diagnosis analysis or ordinary reasoning. These two fields share a common view of abduction as a general process of hypotheses formation. More precisely, abduction is conceived as a kind of reverse explanation where a hypothesis H can be abduced from events E if H is a "good explanation" of E. The paper surveys four known schemes for abduction that can be used in both fields. Its first contribution is a taxonomy of these schemes according to a common semantic framework based on belief revision. Its second contribution is to produce, for each non-trivial scheme, a representation theorem linking its semantic framework to a set of postulates. Its third contribution is to present semantic and axiomatic arguments in favor of one of these schemes, "ordered abduction," which has never been vindicated in the literature.

Key words: Abduction, belief revision, explanation, non-monotonic reasoning

1. Introduction

Abduction was first defined in *epistemology* as a reasoning process leading to form an explanatory hypothesis from given observations, especially in physics. It operates from facts to facts, for instance, when Leverrier postulated the existence of Neptune from the discrepancy between the predicted and the observed trajectory of Uranus. It operates from facts to laws, for instance, when the law of discrete electromagnetic rays was derived from observations of different chemical elements. It operates from laws to theory, for instance, when Newton's theory was conjectured from Kepler's laws and the falling bodies law.

More recently, instances of abduction were given in *artificial intelligence* (*AI*), especially in relation with diagnosis tasks or ordinary reasoning. The first are illustrated by medical diagnosis when a physician guesses the illness which causes some symptoms or by police inquiry when a police officer guesses a criminal from observed clues. The second are found in natural interpretation when an agent tries to reveal his opponent's preferences (or beliefs) through his actions, or in experimental psychology when people try to discover a recurrence rule able to generate a given sequence of numbers.

The aim of the paper is to propose a general definition that suits all the typical instances of abduction as hypotheses formation process, whether in science, in diagnosis or in ordinary reasoning. A common feature of the analysis of abduction is its link to the notion of explanation. Indeed, abduction is widely considered as a kind of reverse explanation. But this feature is not enough to provide a satisfactory definition of abduction since the formal definition of explanation is itself controversial. Moreover, not all (reverse) explanations are acceptable for abduction, but only the "best" or at least the "good" ones (as defined for example in Lipton, 1991).

The most standard way to define abduction is through classical deduction, "classical abduction" from E to H being then defined as reverse classical deduction: $H \subseteq E$. Since such a definition can be shown to be unsatisfactory, a richer approach first consists in considering non-monotonic reasoning (Poole, 1988, 1989). Another proposal consists in introducing a belief revision operation within the antecedent and/or within the consequent of the inference scheme. *Belief revision* is more and more widely accepted as a very powerful and convenient framework to model reasoning. It has been linked with different types of inference, for instance, with non-monotonic reasoning (Kraus et al., 1990) or with confirmation (Zwirn and Zwirn, 1994), and, although more controversially, with conditional reasoning (Stalnaker, 1968).

In a semantic belief revision framework, an agent's initial belief K is revised into a final belief K^*A when the agent receives some message A. Replacing H or E by the respective beliefs K^*H or K^*E leads naturally to three possible alternative schemes to reverse classical deduction. The paper relies on this combinatory heuristic to compare four abduction schemes (including classical abduction), reciprocal of four explanation schemes. Each of these abduction schemes have been independently presented by several AI authors. A recent and systematic analysis of these proposals can be found in Pino-Perez and Uzcátegui (1999). Section 4.3 gives a more detailed analysis of previous works.

The paper compares the four abduction schemes along their common belief revision semantic framework. It discards schemes that allow an agent to abduce hypotheses he should normally not abduce or that prevent him from abducing hypotheses that he could be willing to abduce. The paper proposes further a set of postulates for the last three abduction schemes and proves (for the two original schemes) the representation theorems which link the semantic framework to the set of postulates. Hence, the abduction schemes can be compared through the postulates which are in common and those which differ. The paper finally favors one scheme, "ordered abduction," which has not been vindicated by previous authors. Arguments in favor of this scheme are both semantic and axiomatic and are illustrated by one example.

The paper is organized as follows. The second section recalls the historical background and introduces the formal framework used. The third section defines the four possible abduction schemes in relation with belief revision operations. The fourth section compares the relevance of these schemes through one example and through more theoretical considerations and considers the related works. The fifth section presents the sets of postulates and representation theorems for non-transitive, non-reflexive and ordered abduction. The sixth section compares these sets of postulates and discusses their respective advantages and defaults. A conclusion follows while proofs are given in appendix.

2. Background and Framework

2.1. Abduction Along Peirce

The concept of abduction has been far less studied by the philosophy of science than the concept of explanation. It was first defined by Charles Peirce (1931–1958), and later gained few logical improvements, including Rescher (1978) and Levi (1979). Peirce defines abduction in the following terms: "Abduction is the process of forming an explanatory hypothesis. It is the only logical operation which introduces any new idea." He gave in fact two rather different definitions, a formal one introduced in the treatment of a syllogism and a constructive one stated in the process of belief formation. However, the common idea is to consider abduction as a reverse explanation, in that a proposition abduced from another one must be a good explanation for it.

The first definition of abduction given by Peirce, *abduction*$_1$, stands inside the predicate calculus framework. Consider a syllogism which relates a structural antecedent H (the rule) and a factual antecedent h (the case) to a factual consequent k (the result): $H \wedge h \rightarrow k$. According to Peirce, there are three basic operations between these terms:

- *prediction* links H and h to k,
- *abduction*$_1$ links H and k to h,
- *induction* links couples (h, k) to H.

This analysis is in accordance with the so called "deductive-nomological scheme," on which Hempel (1965) and Popper (1959) relied for building their epistemological theories. Popper put stress on *refutation*, which links $\neg k$ to $\neg H$ or $\neg h$. Hempel proposed a theory of *confirmation*, a concept which encompasses both abduction$_1$ and induction. The last two concepts appear technically as reverse predictions, although induction selects inference to rules (in the context of a case) and abduction$_1$

selects inference to cases (in the context of a rule). But contrary to deduction which preserves the truth value of propositions, abduction (like induction) cannot be logically justified and even falls apparently in the fallacy called "the affirmation of the consequent." Actually, as Peirce clearly states it, abduction is knowledge ampliative.

Although precise, this first definition of abduction encounters two opposite limits:

- Some abductions allowed by this definition are intuitively not admissible because they lead to abnormal assumptions. For instance, if I see that my grass is wet, I would generally not assume that a water bomber has poured the content of its tank on it, though this abduction is allowed by abduction$_1$.
- Some intuitively acceptable abductions are not allowed by this definition since they rely on non-nomological relations between a fact and a possible assumption. For instance, if I see that my grass is wet, though it is a natural assumption to think that my sprinkler is on, I cannot abduce it through abduction$_1$ since the sprinkler may have a breakdown.

These limits are related to the fact that abduction$_1$ is defined within a classical framework, where the notions of "normality" and "exceptions" have no room.

The second definition of abduction given by Peirce, *abduction*$_2$, is a more general mode of inference which is defined in the dynamic context of scientific inquiry. A scientist may learn a surprising fact, which troubles his mental state of "cognitive calm" concerning a given class of phenomena. This surprising fact requires an explanation validated in three reasoning steps:

- abduction$_2$ corresponds to a first step where the scientist formulates some explanatory hypotheses (laws or theories) which, if true, would restore his state of "cognitive calm";
- deduction corresponds to a second step where the scientist infers from the preceding hypotheses some contrasted consequences able to be experimentally tested;
- induction corresponds to a third step where the scientist experiments in order to build degrees of confirmation of the hypotheses, leading eventually to favor one.

A possible reading of this theory is that abduction$_2$ belongs to the context of discovery, the context of justification being reserved to deduction and induction. This could imply that a logical analysis of abduction$_2$ is impossible since heuristics is not a purely logical process. Furthermore, even if logification is relevant, abduction$_2$ would not even be an inference because it does not lead to "conclusions" but to mere "candidates to belief." However, according to most of the Peirce's analysts, a logic of abduction$_2$ can be proposed since not every hypothesis is admissible as a good candidate for belief: even if not accepted, abduced hypotheses result from a selection of the explanations that can be "seriously considered" for further acceptance. This requires to suggest a logical criterion for this selection, a task that was not achievable by Peirce at his time.

Actually, abduction_2 is not incompatible with abduction_1. It can rather be thought as a more general inference which associates abduction with two constraints:

(i) abduced hypotheses must "explain" the facts under consideration, eventually in a given context;
(ii) abduced hypotheses must be "good candidates to belief".

A motto which seems to encompass both constraints and is often endorsed by abduction theorists is that abduction is *inference to the best explanation* (see Harman, 1978; Thagard, 1978; Lipton, 1991 for a detailed analysis of this concept and van Fraassen, 1980, for a critical appraisal of its use in favor of scientific realism). However, the notion of "best" explanation is too demanding since abduction may select several candidates to belief. Hence, the guideline for a further analysis will be that abduction is simply *inference to a good explanation*. Usually, an explanation scheme appears as a "forward inference" which involves a proposition A (for instance, a case) explaining a proposition B (for instance, a result), eventually in some context (for instance, a law). Conversely, an abduction scheme can be viewed as a "backward inference" from the explanandum B to the explanans A, a condition realized by both abduction_1 and abduction_2.

2.2. Belief Revision

Two logical frameworks are usually considered. The syntactic framework is defined by a formal language L built by use of a finite set of propositions $\{a, b, c, \dots\}$ closed under the connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$(disjunction) and $\rightarrow$ (implication). Let T and $\perp$ be the two constants truth and falsity. Let $\vdash$ be the symbol of the meta-level deduction operation. The semantic (set-theoretic) framework is defined on a (finite) set of possible worlds with the set operations: - (complementation), $\cap$ (intersection), $\cup$ (union) and $\subseteq$ (inclusion). Let $A, B, C, \dots$ be events, defined as subsets of worlds. Let W and Ø be respectively the full set and the empty set.

The two frameworks are isomorphic in a propositional language with a finite number of propositional letters, with the following correspondences. First, to each proposition x is associated an event X, i.e. the set of worlds where the proposition is true. Second, the symbols: $\neg$, $\wedge$, $\vee$, $\vdash$ correspond to the symbols -, $\cap$, $\cup$, $\subseteq$. Since the latter framework is computationally more convenient, it will be favored for the exposition of abduction schemes as well as for the proof of representation theorems. However, the terms 'propositions' and 'events' will be used one for the other. Adaptation to a propositional framework (needing just a rewriting) and generalization to an infinite number of possible worlds or an infinite language is left to further work.

Belief revision is a belief change operation $*$ which relates an initial agent's belief K and a message A (which may contradict the initial belief) to a final belief K^*A. Beliefs K and K^*A are assumed to be subsets of W. Contrary to W, K is assumed to evolve when the agent makes new observations or receives new information from other agents. The basic postulate of belief revision is that the message has an epistemic priority over the initial belief of an agent, due to more direct observations or to more reliable sources. This postulate is shared by abductive reasoning.

In the syntactic framework of propositional logic, it is usual to introduce explicitly a *background theory* Σ. Such a theory considers some generic beliefs endorsed by the agent. In the belief revision framework, such beliefs will be considered as embedded partially in W and partially in K. Beliefs inside Σ which are fixed are directly incorporated as constraints in the set W. Beliefs inside Σ which could change, are included in the agent's belief K which contains generic beliefs (i.e., laws) as well as specific ones (i.e., facts) and which is revised when something changes.

Belief revision was duly axiomatized by Alchourron et al. (1985) according to the following postulates:

A1. Consistency
If $K \neq \emptyset$ and $A \neq \emptyset$ then $K^*A \neq \emptyset$

A2. Success
$K^*A \subseteq A$

A3. Conservation
If $K \subseteq A$ then $K^*A = K$

A3′. Weak Conservation
$K^*T = K$

A4. Sub-Expansion
$(K^*A) \cap B \subseteq K^*(A \cap B)$

A4′. Inclusion
$K \cap A \subseteq K^*A$

A5. Super-Expansion
If $(K^*A) \cap B \neq \emptyset$ then $K^*(A \cap B) \subseteq (K^*A) \cap B$

A5′. Preservation
If $K \cap A \neq \emptyset$ then $K^*A \subseteq K \cap A$

A45. Right Distributivity
$K^*(A \cup B) = K^*A$ or K^*B or $K^*(A) \cup K^*(B)$

It is possible to prove that the following set of postulates are equivalent:

- {A1, A2, A3, A4, A5}
- {A1, A2, A3′, A4, A5}
- {A1, A2, A4, A4′, A5, A5′}
- {A1, A2, A3, A45}
- {A1, A2, A3′, A45}

Belief revision rules can be associated with the set of postulates by a representation theorem (Alchourron et al., 1985). Consider a preference relation represented by a total preorder $\leq_K$ on W indexed on a subset K of W. It is decomposed as usually into $<_K$ (by $w <_K w'$ iff $w \leq_K w'$ and not $w' \leq_K w$) and $=_K$ (by $w =_K w'$ if $w \leq_K w'$ and $w' \leq_K w$). These relations are assumed to fulfill two properties:

(i) $w' \in K$ and $w'' \in K \Rightarrow w' =_K w''$,
(ii) $w' \in K$ and $w'' \notin K \Rightarrow w' <_K w''$.

It defines a ranking of the worlds of W, which can be represented by a system of concentric "spheres" around K. These embedded spheres cut up coronas between two successive ones. The more distant coronas correspond to the subsets of less preferred worlds. The minimal worlds of an event A (called the 'preferred' or the 'normal' part of A) are now defined by:

$$\mathrm{Min}_K(A) = \{w \in A : \forall\, w' \in A,\, w' <_K w \text{ is false}\}$$

The representation theorem states that the revised belief is the set of the minimal worlds belonging to the message (the "preferred" part of the message):

$$K^*A = \mathrm{Min}_K(A)$$

It means that the final belief is the intersection between A and the sphere of the closest worlds to K which has a non-empty intersection with A (see Figure 1). **Q1**

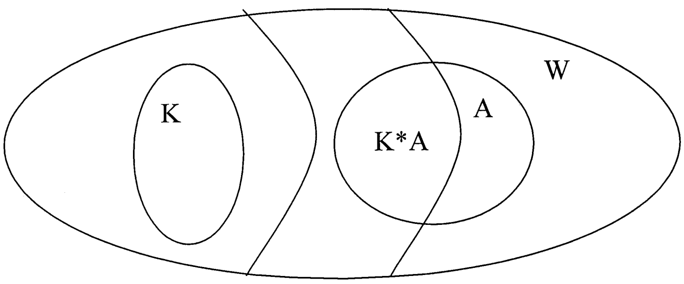


Figure 1.

The preference relation $\leq_K$, which is specific of one agent's "epistemic state" (Darwiche and Pearl, 1997), is a more complete description of the agent's total belief than K is and defines all what is needed to achieve his belief revision process.

2.3. Non-Monotonic Reasoning

Non-monotonic inference weakens the usual operation of classical deduction in order to reflect rules of common reasoning in the context of proof. These rules do not preserve anymore the truth value of the propositions. More precisely, a non-monotonic inference $|\sim$ states that $A|\sim B$ means "if A, normally B" or "if A is considered as true, then B is accepted." This kind of inference is non-monotonic since adding a new premise A' to A does not necessarily preserve the initial conclusion B.

Non-monotonic inference was duly axiomatized by Kraus et al. (1990), who introduced a set of postulates corresponding to "preferential" non-monotonic inference and Lehmann and Magidor (1992), who introduced a set of postulates corresponding to "rational" non-monotonic inference, strictly stronger than the first one. Only the second will be used in Section 5.1. The corresponding postulates are the following:

C0. Left Logical Equivalence
If $A \equiv B$ and $A|\sim C$ then $B|\sim C$

C1. Right Weakening
If $A \subseteq B$ and $C|\sim A$ then $C|\sim B$

C2. Reflexivity
$A|\sim A$

C3. Right And
If $A|\sim B$ and $A|\sim C$ then $A|\sim B \cap C$

C4. Left Or
If $A|\sim C$ and $B|\sim C$ then $A \cup B|\sim C$

C5. Consistency Preservation
If $A|\sim \emptyset$ then $A \equiv \emptyset$

C6. Cautious Monotony
If $A|\sim B$ and $A|\sim C$ then $A \cap B|\sim C$

C7. Rational Monotony
If (not $(A|\sim -B)$ and $A|\sim C$) then $A \cap B|\sim C$

It is possible to prove that the following set of postulates are equivalent:

– {C0, C1, C2, C3, C4, C6, C7},
– {C0, C1, C2, C3, C4, C5, C7}.

A representation theorem was given (Lehmann and Magidor, 1992). The authors make a technical distinction between a set of “states,” designing possible states of affairs, and a set of “worlds,” designing the truth values assigned to propositions. A given state may correspond to a subset of worlds. According to Makinson (1993), this distinction can be avoided by assuming that there exists a one-to-one correspondence between “states” and “worlds,” i.e. elements of W.

Consider then a preference relation defined by a partial preorder $\leq$ on W, which admits the complementary relation $<$ on W. The minimal worlds of an event A, denoted Min(A), are defined by:

$$\mathrm{Min}(A) = \{w \in A : \forall w' \in A,\ w' < w \text{ is false}\}$$

Then a rational non monotonic inference $A|\!\sim B$ holds when, for every model satisfying the preference relation the consequent B is true in every minimal world satisfying A:

$$A|\!\sim B \quad \text{iff } \mathrm{Min}(A) \in B$$

Let us consider again the relation $<_K$ related to K (see Section 2.2) and define the non-monotonic inference relation by:

$$\begin{aligned} A|\!\sim_K B \quad &\text{iff } \mathrm{Min}_K(A) \subseteq B, \text{ with } \mathrm{Min}_K(A) \\ &= \{w \in A : \forall w' \in A,\ w' <_K w \text{ is false}\} \end{aligned}$$

The following correspondence rule between rational non-monotonic inference and belief revision has been proved (Gärdenfors and Makinson, 1991):

$$A|\!\sim_K B \quad \text{iff } K * A \subseteq B$$

The initial belief K acts as a parameter for specifying partially the preference relation underlying the non-monotonic inference:

$$K = \cap B : W|\!\sim_K B$$

3. Four Abduction Schemes

3.1. Abduction as Belief Revision

Abduction has already been linked to non-monotonic reasoning and belief revision in different ways (see Section 4.3). The intuitive arguments for relating them are the following:

– The abduction process requires some belief change operation to occur. Indeed, abduction relates an initial belief and a new observation to a final belief changed through the abduction process in order to include a new hypothesis.

– Abduction is, as belief revision, ampliative and non monotonic. First, as previously noticed, abduction leads to infer hypotheses that cannot be classically deduced from the given facts. Second, when a hypothesis is a good explanation of some facts, it does not mean that it is a good explanation of these facts jointly to some other facts.

However, one cannot consider that belief revision or non-monotonic inference are directly relevant theories of abductive reasoning. Such a "direct equivalence" would state that a hypothesis H is abduced from facts E either iff $K{*}E \subseteq H$ (belief revision) or iff $E \mid\!\sim H$ (non-monotonic inference). Such a thesis has to be rejected for two reasons. First, this use of belief revision or of non-monotonic reasoning introduces a direct inference from facts to hypotheses. However, as considered in this paper, abduced hypotheses have to be an explanation of facts and need to entail them in some way. Second, hypotheses implied by a belief revision operation or resulting from a non-monotonic inference are "accepted" by the agent and integrated in his final belief. However, as considered in this paper, abduced hypotheses are only "serious candidates" for acceptation and their acceptance depends on further tests between them.

In this paper, belief revision will be favored in order to formalize abduction. Non-monotonic reasoning is only used indirectly since belief revision naturally introduces some non-monotonicity. A good logical definition of abduction must state which belief revision operations are adequately involved when selecting hypotheses which are "seriously considered" without being necessarily accepted. The problem considered is then to propose a complete taxonomy of the possible relations between abduction and belief revision.

In the preceding semantic framework, the facts (propositions that are true) as well as the hypotheses (propositions to be assumed) will be called events. When deduction is considered, it will always be interpreted as classical deduction. As concerns explanation, a hypothesis is said to be an "explanation" of a fact when at least some subset of the fact is deductively implied by some subset of the hypothesis. In order to deal with abduction, a generic operators is added: $\|\!\to$. By definition, $E \|\!\to H$ means that the hypothesis H is abduced from event E, or equivalently that the event E is (well) explained by the hypothesis H. Some cases are trivial, for instance, when E is deduced from K, but all definitions and postulates apply.

3.2. Formal Definition of the Abduction Schemes

In the following, the arrow ($\to$) used generically for all forms of abduction will be replaced by different signs for each specific abduction scheme in order to relate easily the different schemes. All definitions of abduction schemes are stated given a revision operation * and a background knowledge K. Four abduction schemes are defined by using all possible combinations of the revision operation acting (or not) on facts E and hypothesis H. Other possible schemes could be imagined by considering for instance the negation of propositions E and H. But the present paper will be focused only on the criteria involving one condition, which are the simplest ones, this choice being common to the other works concerning the link of abduction with belief revision in the literature (see Section 4.3).

The basic scheme usually considered is *classical abduction* (reverse classical explanation) defined by the following condition (where ||- should *not* be interpreted as semantic deduction):

$$E \,\|{=}\, H \quad \text{iff} \;\; H \subseteq E$$

(The label "classical" just refers to classical logic where no belief revision operation is involved.) This abduction scheme is the most straightforward conception of an inference to a good explanation.

The second scheme defines *non-transitive abduction* (reverse non-transitive explanation) by the following condition:

$$E \,\|{\sim}\, H \quad \text{iff} \;\; K * H \subseteq E$$

(The label "non-transitive" is favored over the label "non-monotonic" since other explanation and abduction schemes will be non-monotonic while this one is the only one to be non-transitive). This abduction scheme is logically weaker than the previous one. It states that abduction is not reverse deduction but rather reverse belief revision (hence reverse non-monotonic inference): one abduces a hypothesis from a fact if one would have added this fact to one's belief after having revised initial belief by the hypothesis (or equivalently if one infers non-monotonically the fact from the hypothesis). That means that the "normal" part of the hypothesis H must imply the fact E.

A third scheme defines *non-reflexive abduction* (reverse non-reflexive explanation), by the following condition (including for technical reasons that a contradiction cannot be abduced):

$$E \,\|{\prec}\, H \quad \text{iff} \;\; \emptyset \neq H \subseteq K * E$$

(This abduction scheme is called non-reflexive since it is the only one with that property.) It is logically stronger than classical abduction. It states that one abduces a hypothesis from a fact if it explains the revised fact deductively. That means that the hypothesis H must imply the "normal" part of the fact E.

The last scheme defines *ordered abduction* (reverse ordered explanation) by the following condition:

$$E \,\|{\approx}\, H \quad \text{iff} \;\; \emptyset \neq K * H \subseteq K * E$$

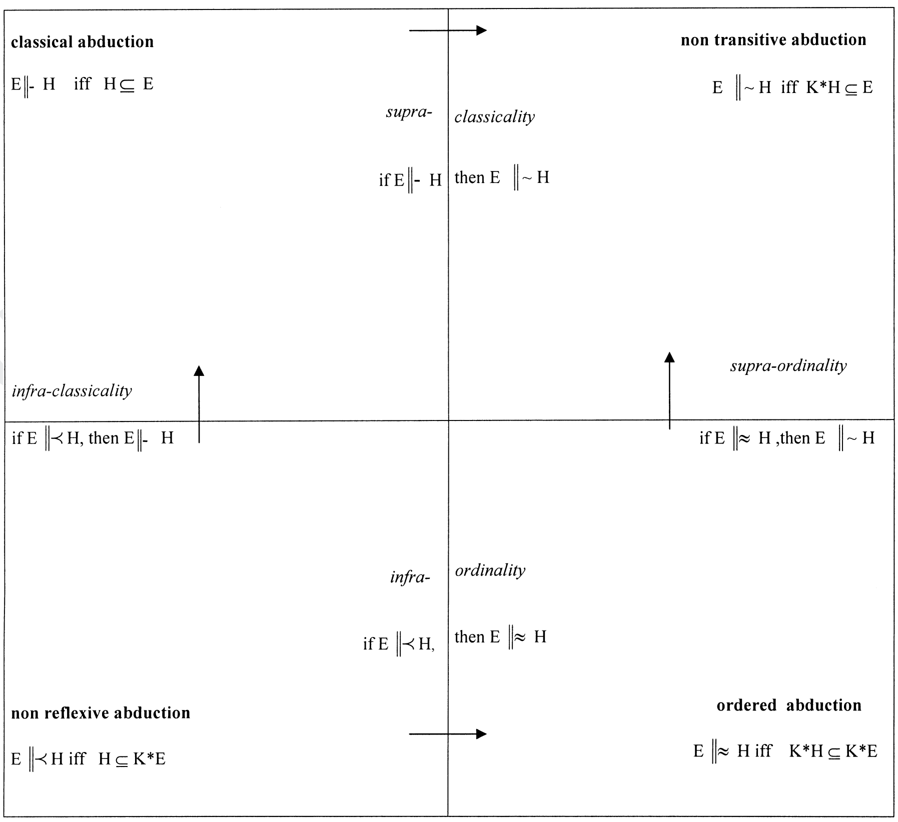
(The term ordered has been chosen since the binary relation is now reflexive and transitive and hence is a pre-order; it is the only abduction scheme to satisfy these properties except for classical abduction). This abduction scheme is stronger than non-transitive abduction, weaker than non-reflexive abduction, and cannot be compared to classical abduction. It considers that antecedent and consequent are both contextualized by prior belief and relies on the fact that the belief revised by the hypothesis would logically imply the belief revised by the fact. That means that the "normal" part of the hypothesis H must imply the "normal" part of the fact E.

3.3. A SYNTHETIC TABLE

In Table I, the four abduction operations are located in the periphery. Moreover, the
Q2 relations of implication between them are denoted in the following way:

Table I.

Abductive Logics in a Belief Revision Framework



- infra (supra)-classicality compares any abduction scheme to classical abduction:

  infra–classicality: if $E \Vert\!\!\rightarrow H$, then $E \Vert\!\!\rightarrow H$
  supra–classicality: if $E \Vert\!\!- H$, then $E \Vert\!\!\rightarrow H$

- infra (supra)-ordinality compares any abduction scheme to ordered abduction:

  infra–ordinality: if $E \Vert\!\!\rightarrow H$, then $E \Vert\!\!\approx H$
  supra–classicality: if $E \Vert\!\!\approx H$, then $E \Vert\!\!\rightarrow H$

4. Semantic Comparison of Abduction Schemes

4.1. One Example

As a simple example, consider the fact that 'the grass of my garden is wet'. Several abductions can be made, each abduction implying the weaker ones. A classical abduction could be that 'a water bomber poured water on it' (assuming that we can safely deduce that if it was the case, the grass will certainly be wet). An instance of non-transitive abduction (the normal part of the hypothesis H must imply the fact E) could be that 'a sudden overflow of the near river happened' because, however improbable it is, if such an overflow was to occur, it would normally flood my garden. Under non-reflexive abduction (the hypothesis H must imply the normal part of the fact E), one may infer that 'it rained on my garden' because it is an usual situation for wet grass to have received rain. An ordered abduction (the normal part of the hypothesis H must imply the normal part of the fact E) could allow the hypothesis that 'the sprinkler is on' because on one hand, if it is the case that if the sprinkler is on, normally the grass is wet, and on the other hand, it is often the case that the grass is wet because the sprinkler has been put on.

Classical and non-reflexive abductions lead to infer hypotheses whose occurrence seem to imply the fact that the grass is wet, while ordered and non-transitive abductions rely on hypotheses whose occurrence tolerates exceptions to that fact: the sprinkler could be broken and the overflow of the river could be too small to wet the grass. Classical and non-transitive abductions may be discarded since they allow to infer quite implausible hypotheses. On the contrary, although non-reflexive abduction seems relevant, it would be too restrictive to allow the sprinkler hypothesis which allows to infer that the grass is wet only through a non-monotonic inference. Ordered abduction is the only scheme allowing both to abduce the rain hypothesis and the sprinkler hypothesis. Hence, it seems to be the more relevant scheme.

Remark. This example makes clear the link of abduction to explanation. Considering ordered abduction, from the fact that 'the grass is wet', one can abduce that 'it rained' or that 'the sprinkler is on', because both hypotheses directly explain the wet grass. However, from the fact that 'the grass is wet and the sprinkler is out', one cannot abduce that it rained because this hypothesis does not explain the whole fact since it does not explain that the sprinkler is out. This may seem to contradict a possible intuition, since discarding the rival active sprinkler assumption should reinforce the rain assumption. But this intuition, if one grants it, comes from a confusion between the conjunction 'the grass is wet and the sprinkler is out' with the fact 'the grass is wet' inside an initial belief including the fact that 'the sprinkler is out'. In the last case, it would be necessary first to revise from the fact that the sprinkler is out and then to abduce from the fact that the grass is wet the hypothesis that it rained. But this involves iterated change not considered here.

4.2. General Discussion

Classical abduction is inadequate for two reasons. It is too weak because a fact can be deduced from a lot of "strange" hypotheses since any subset of the antecedent is an abduced consequent. But all sufficient conditions can not be considered as "good explanations" of a derived fact. For instance, if I see something flying in the sky, I can abduce—but in a strange way—that it is a flying saucer since a flying saucer always flies. It is also too strict because a good explanation of a fact is not always a hypothesis from which this fact can be logically derived. In a lot of situations, no interesting deductive explanation (by sufficient conditions) may be available. For instance, if I see something flying in the sky, I cannot abduce—contrary to intuition—that this is a bird because if many birds fly, not all birds fly (penguins, ostriches).

Non-transitive abduction takes into consideration the fact that deductive explanations are not always available and that most good explanations are often non-monotonic inferences that can be defeated by counterexamples. It addresses correctly the second default of the classical abduction scheme, by accepting some good candidates that classical abduction would have rejected. For instance, it allows the abduction that some flying object in the sky could be a bird because normally a bird flies. But, it does not address its first default: it is still too weak and would lead to accept a lot of bad candidates for abduction. In particular, it does not discard the abduction about the flying saucer.

Non-reflexive abduction and ordered abduction need a more precise discussion. Both abductions (contrary to classical and non-transitive abductions) address correctly the first default of classical abduction. They concentrate on the best explanation of a fact by ruling out "abnormal" hypotheses. For instance, they discard the abduction about the flying saucer. Technically, this is due to the fact that, when receiving a new piece of information E, the initial belief K is revised according to message E before proceeding to abduction. However, two arguments in favor of non-reflexive abduction will be successively refuted.

The *first argument* states that non-reflexive abduction corresponds to a deductive explanation contrary to ordered abduction (observe that the same argument can be stated for classical abduction with regard to non-transitive abduction). Consider a couple of events (H, E) such that $K * H \subseteq K * E$ but $H \not\subseteq K * E$. It is possible to abduce H by ordered abduction but not by non-reflexive abduction. Let us call H' the hypothesis $K * H$. It is possible to abduce H' by non-reflexive abduction (and of course by ordered abduction too). Now, why should an agent abduce H which is not as good as an explanation as H', since E is deductively implied by H'? One could think that non-reflexive abduction which allows the agent to abduce H' and not H is a better type of abduction than ordered abduction which allows him to abduce also H. For instance, if I see a flying object in the sky, the hypothesis that it is a "flying bird" (a "non-penguin" bird) could be considered as a better abduction than the hypothesis that it is just a bird (which is not selected through non-reflexive abduction).

However, this argument is theoretically but not practically acceptable and does not sustain non-reflexive abduction for the following reason (applying to classical abduction too). First, for a finite set of worlds (or a finite set of propositions), it seems possible to state explicitly all exceptions to any given rule. But such a way to deal with the problem quickly leads to a number of cases which prevents any real treatment for a human reasoning agent because of the combinatorial explosion that arises. More generally, the relevance of non-monotonicity for ordinary (and even scientific) reasoning has to be seriously taken into account. The starting point of non-monotonic logic is that the set of possible worlds handled by a reasoning agent is generally not refined enough to establish deductive relations between empirical events. The proposition "if A then B" is relative to a set of empirical conditions or "provisos" and the set of these provisos is generally intractable or even infinite (Hempel, 1988). For instance (Goodman, 1955), if you see a lighted match, you can explain it by the fact that somebody scratched it, but it is not enough because you have also to assume that the match was not wet, that there was no wind and so on. Hence, ordinary reasoning is better represented by propositions such as "if A then normally B". The set of possible worlds considered by the modeler to give a semantic interpretation to this kind of propositions (in terms of "minimal worlds") is necessary finer than the set of possible worlds considered by the agent. Hence, when considering abduction, it is a philosophical fallacy to recommend that the agent should use this finer set of worlds to perform his reasoning task. The proposition $H' = K * H$ will generally not be expressible in the vocabulary used by the agent (or even the modeler) who is constrained to use H (the general hypothesis alone).

Remark. The standard "bird" example (like all examples in "small worlds") is a bit misleading because it is too simple. Speaking of "flying birds" treats H' as the conjunction of H and E. It is true that if a hypothesis H is a non-monotonic explanation of E: $H \mid\sim E$ (or equivalently $K * H \subseteq E$), then the conjunction of E and H will be a deductive explanation of E (this is even true for any hypothesis H compatible with E). But it is not in the spirit of abduction to abduce from E the conjunction of E and of another hypothesis H. In the scientific work as well as in the usual life, due to the limitations of language, it is generally impossible to express a hypothesis which actually entails the observed event from a purely deductive point of view. By requiring that the hypothesis should deductively imply the normal cases of the fact, non-reflexive abduction prevents from considering non-monotonic relations between an explanans and an explanandum, and it can often be impossible to find an interesting hypothesis which satisfies this requirement.

The *second argument* states that non-reflexive abduction is not reflexive contrary to ordered abduction. Reflexivity, which means that is always possible to abduce a fact from itself, is not an intuitively desirable feature since one does not gain anything from a so poor abduction.

But this argument points only toward an overall limit of the present framework, common to most qualitative frameworks: it does not allow to compare the degrees of "explanatory power" of different hypothesis. One cannot argue that non-reflexive abduction is the proper answer to formalize this notion, since if $E = K * E$ (the normal part of E includes all worlds in E) then non-reflexive abduction allows also to abduce E from itself.

Ordered abduction will then be favored as the only realistic type of abduction for ordinary or scientific reasoning. It validates the idea that an explanation may be a non-monotonic relation between hypotheses and facts, but conversely accepts the restriction that good explanations of an event are those which validate only its normal ways to be true, i.e. its preferred interpretations. It simultaneously allows the "bird" hypothesis and rules out the flying saucer. This seems to a be a good compromise between the two defaults of classical abduction. An interesting consequence of this conclusion is that abduction cannot be simply defined by the inversion of a consequence relation which would describe "good explanations": neither deduction nor non-monotonic inference are adequate definitions of good explanations.

Nevertheless, it is possible to lessen the gap between ordered and non-reflexive abduction if one accepts to consider that, in a typical abduction situation, an agent would only hesitate between a fixed set of exclusive abducible hypotheses. These exclusive hypotheses are for instance the set of possible answers to one question (Levi, 1979), the possible diseases of a patient or the possible murderers for a crime (like in the game of *Cluedo*). Hence, the agent does not consider all possible subsets of the set of possible worlds W but the cells of a partition of W, belonging to $W' \subset 2^W$. From the agent's point of view, the reasoning task is performed within W', and the result of an abduction is always a single cell. In that case, the definitions of ordered and of non-reflexive abductions collapse since $K * H = H$ for any hypothesis H. Such a situation is in accordance with the previous remark: the set of possible hypotheses within which the abductive task is *de facto* performed is not refined enough to allow the agent to proceed to deductive explanations of an empirical phenomenon.

In other respects, abduction is a dynamic process in the spirit of the Peircean theory of abduction$_2$ and may lead to more and more precise abduced hypotheses, converging eventually towards a deductive explanation. When receiving more and more information about various cases, the agent may revise the preorder between worlds by distinguishing worlds which were initially in a same corona. In the limit, each world can be singularized. In this case, the definition of ordered and non-reflexive abductions collapse again since H is a singleton. Such an asymptotic situation is again in accordance with the preceding remark: if the set of possible worlds is refined enough, ordered abduction may converge asymptotically towards non-reflexive abduction.

4.3. Related Works

This section considers the works which are directly related to the present paper, i.e. the formulation of purely logical definitions of abduction in relation with belief revision. It does not consider other works, dealing for instance with direct relation between abduction and non-monotonic reasoning (Poole, 1988, 1989).

Classical abduction can be associated with the axiomatic system proposed by Flach (1996) under the name of "explanatory induction," as shown by Pino-Pérez and Uzcátegui (1999, Section 5).

Non-transitive abduction is proposed by Boutilier and Becher (1995) under the name of "predictive explanation." It is introduced by Pino-Pérez and Uzcátegui (1999) under the label "epistemic explanation" in relation with belief revision.

Non-reflexive abduction gives a belief revision semantics to the criterion proposed by Cialdea Mayer and Pirri (1996). It is introduced by Pino-Pérez and Uzcátegui (1999) under the label "causal explanation" in relation with non-monotonic inference. The heuristic they adopt consists in relating abduction to non-monotonic reasoning in the same spirit that we relate abduction to belief revision. More precisely, they associate to abduction, denoted $E \rhd H$, an inference relation, denoted $E \mid\sim_{ab} F$, by the following relation:

$$E \mid\sim_{ab} F \text{ if (if } E \rhd H \text{ then } H \subseteq F)$$

They impose to $\mid\sim_{ab}$ to satisfy several postulates of the non-monotonic inference of Kraus, Lehmann & Magidor (1990) and they look for the corresponding postulates for $\rhd$. They define stronger and stronger set of postulates with more postulates till reaching causal explanation with all postulates. The last is shown to satisfy:

$$E \rhd H \quad \text{iff (if } E \mid\sim_{ab} F \text{ then } H \subseteq F)$$

It is easy to see that it corresponds precisely to non-reflexive abduction.

Ordered abduction is also considered by Pino-Pérez and Uzcátegui (1999) under the label "strong epistemic explanation" in relation with belief revision. In fact, they discard it in favor of non-reflexive abduction by using two types of arguments. First, they notice that in some cases, ordered explanations "are not even explanations," in the sense that the observation E may not follow deductively from the abduced hypothesis H. However, the present paper vindicates the idea that good explanations are not necessarily deductive and even, that they are generally not. Second, they follow their own heuristic described before. But they do not give strong arguments in its favor. In fact, the same heuristic leads to ordered abduction if the inference relation $|\!\sim_{ab}$ is defined by:

$$E|\!\sim_{ab}F \text{ if (if } E \rhd H \text{ then } H|\!\sim F)$$

while $|\!\sim$ satisfies the KLM postulates. The reverse relation is then:

$$E \rhd H \quad \text{iff (if } E|\!\sim_{ab}F \text{ then } H|\!\sim_{ab}F)$$

5. Postulates and Representation Theorems

5.1. Non-Transitive Abduction

Since non-transitive abduction has been shown to be equivalent to reverse rational non-monotonic inference, it is enough to reverse the *postulates* of rational non-monotonic inference. Consistency preservation is not considered since nothing can be abduced from the empty set. The remaining *postulates* are the following:

B1. Reflexivity
If $H \neq \emptyset$ then $H \parallel\!\sim H$

B5. Right Or
If $(E \parallel\!\sim H) \wedge (E \parallel\!\sim G)$ then $E \parallel\!\sim G \cup H$

B9. Left And
If $(E \parallel\!\sim H) \wedge (F \parallel\!\sim H)$ then $E \cap F \parallel\!\sim H$

B10. Left Weakening
If $(E \parallel\!\sim H) \wedge (E \subseteq F)$ then $F \parallel\!\sim H$

B11. Rational Right Strengthening
If $(E \parallel\!\sim H) \wedge$ not $(-F \parallel\!\sim H)$ then $E \parallel\!\sim F \cap H$

B1 means that every non-contradictory hypothesis is abduced from itself. B5 states that the disjunction of two hypotheses abduced from an event is also abduced from this event while B9 states that one hypothesis abduced from two events is abduced from the conjunction of these events. B10 asserts that if a hypothesis is abduced from an event which implies another one, it is also abduced from the last one. Finally, B11 asserts that if from an event one abduces a hypothesis which is not abduced from the negation of another event, the conjunction of the hypothesis and of the second even can be abduced from the first event.

No original *representation theorem* is needed.

5.2. Non-Reflexive Abduction

The proposed *postulates* are the following:

B0. Non-Contradiction
If $E \parallel\!\prec H$ then $H \neq \emptyset$

B1′. Pointwise Reflexivity
w $\parallel\!\prec$ w

B2. Strong Left Or
If $(E \parallel\!\prec F) \wedge (G \parallel\!\prec H)$ then $(E \cup G) \parallel\!\prec F \vee (E \cup G) \parallel\!\prec H$

B3. Infra Classicality
If $E \parallel\!\prec H$ then $H \subseteq E$

B4. Right Strengthening
If $(E \parallel\!\prec H) \wedge (G \subseteq H)$ then $E \parallel\!\prec G$

B5. Right Or
If $(E \parallel\!\prec H) \wedge (E \parallel\!\prec G)$ then $E \parallel\!\prec G \cup H$

B6. Weak Monotony
If $(E \parallel\!\prec H) \wedge (H \subseteq F)$ then $E \cap F \parallel\!\prec H$

B7. Weak Cut
If $(E \parallel\!\prec G) \wedge (G \subseteq F) \wedge ((E \cap F) \parallel\!\prec H)$ then $E \parallel\!\prec H$

B0 says that a contradiction can never be abduced and B1′ states that every world is always self abduced. B2 says that if two hypotheses are respectively abduced from two events, then one of them at least is abduced from the disjunction of the events. B3 means that one abduces only a hypothesis from which the event can be classically deduced. Concerning the conclusion side, B4 says that it is always possible to strengthen an abduced hypothesis and B5 that it is always possible to abduce the disjunction of two abduced hypotheses. Concerning the premise side, B6 means that it is always possible to add to the premises of an abduction any consequence of the hypothesis while B7, in the opposite, means that it is always possible to cut among the premises of an abduction on the condition that one of the premises or an antecedent of it can be abduced from another premise.

The corresponding *representation theorem* states:

THEOREM 5.1. *Let* $*$ *be a revision function satisfying AGM set of postulates* $\mathbf{A} = \{A1, A2, A3, A4, A5\}$, *then an inference relation* $\parallel\!\prec$ *defined* *according to* $(E \parallel\!\prec H) \equiv (\emptyset \neq H \subseteq K * E)$ *respects the set of postulates* $\mathbf{B}_{\mathrm{NR}} = \{\mathrm{B0, B1', B2, B3, B4, B5, B6, B7}\}$ *and therefore it is a non-reflexive abductive inference relation.*

Conversely, let $\parallel\!\prec$ *be a non-reflexive inference relation satisfying the set of postulates* $\mathbf{B}_{\mathrm{NR}} = \{\mathrm{B0, B1', B2, B3, B4, B5, B6, B7}\}$. *Then the operation* * *defined by* $K * E = \cup H\colon E \parallel\!\prec H$ *(union of all events abducted from* E*) where* $K = K * T$, *respects the set of postulates* $\mathbf{A} = \{\mathrm{A1, A2, A3, A4, A5}\}$ *and therefore it is a revision function. Moreover,* $(E \parallel\!\prec H) \equiv (\emptyset \neq H \subseteq K * E)$ *and* $K * E = \{w\colon E \parallel\!\prec w\}$.

The proof is given in Appendix A.

Remark. Notice that in this case, $K * E$ can be seen as the set of all events abduced from E.

5.3. Ordered Abduction

The proposed *postulates* are the following:

B1. Reflexivity
If $H \neq \emptyset$ then $H \parallel\!\approx H$

B3′. Weak Infra Classicality
If $E \parallel\!\approx H$ then $E \cap H \neq \emptyset$

B4′. Weak Right Strengthening
If $(E \parallel\!\approx H) \wedge (\emptyset \neq G \subseteq H)$ then $(E \parallel\!\approx G) \vee (E \cap (-G)) \parallel\!\approx E)$

B5. Right Or
If $(E \parallel\!\approx H) \wedge (E \parallel\!\approx G)$ then $E \parallel\!\approx G \cup H$

B6. Weak Monotony
If $(E \parallel\!\approx H) \wedge (H \subseteq F)$ then $E \cap F \parallel\!\approx H$

B8. Transitivity
If $(E \parallel\!\approx F) \wedge (F \parallel\!\approx G)$ then $E \parallel\!\approx G$

B9. Left And
If $(E \parallel\!\approx H) \wedge (F \parallel\!\approx H)$ then $(E \cap F) \parallel\!\approx H$

B1 is a strengthening of B0, every hypothesis being here self abduced. B3′ restricts infra classicality to the fact that abduced hypotheses are at least not contradictory with the event considered. B4′ weakens B4 and states that either it is possible to strengthen an abduced hypothesis from a given premise, or that premise can be abduced from the conjunction of itself and the negation of the strengthened hypothesis. B5 and B6 are as before. B8 states a classical transitivity property. Finally, B9 says that abduction is preserved by the conjunction of premises from which the same hypothesis can be abduced.

The corresponding *representation theorem* is given below: Q3

THEOREM 5.2. *Let * be a revision function satisfying AGM set of postulates* $\mathbf{A} = \{A1, A2, A3, A4, A5\}$, *then an inference relation* $\|\approx$ *defined according to* $(E \|\approx H) \equiv (\emptyset \neq K * H \subseteq K * E)$ *respects the set of postulates* $\mathbf{Bo}_{\mathrm{R}} = \{B1, B3', B4', B5, B6, B8, B9\}$ *and therefore it is an ordered abductive inference relation.*

Conversely, let $\|\approx$ *be an ordered inference relation satisfying the set of postulates* $\mathbf{Bo}_{\mathrm{R}} = \{B1, B3', B4', B5, B6, B8, B9\}$. *Then the operation * defined by* $K * E = \cap H : H \|\approx E$ *(intersection of all events from which E can be abduced) and where* $K = K * T$, *respects the set of postulates* $\mathbf{A} = \{A1, A2, A3, A4, A5\}$, *and therefore it is a revision function. Moreover,* $(E \|\approx H) \equiv (\emptyset \neq K * H \subseteq K * E)$ *and* $K * E = \{w : E \|\approx w\}$.

The proof is given in Appendix B.

Remark. notice that in this case, $K * E$ can be seen as the common part of all events from which E can be abduced.

6. Syntactic Comparison of Abduction Schemes

6.1. Summary of Postulates

Table II shows the logical links between the three sets of postulates, discarding classical abduction. The postulates entering in their definition are presented in bold characters. The derivation of other postulates is proved in the Appendix. Q4

Table II.

	Non-reflexive abduction	Ordered abduction	Non-transitive abduction
B0: Non-contradiction	**Yes**	Yes	Yes
B1′: Pointwise Reflexivity	**Yes**	Yes	Yes
B3′: Weak Infra Classicality	Yes	**Yes**	Yes
B4′: Weak Right Strengthening	Yes	**Yes**	Yes
B5: Right Or	**Yes**	**Yes**	**Yes**
B6: Weak Monotony	**Yes**	**Yes**	Yes
B7: Weak Cut	**Yes**	Yes	Yes
B9: Left And	Yes	**Yes**	**Yes**
B1: Reflexivity	No	**Yes**	**Yes**
B3: Infra Classicality	**Yes**	No	No
B4: Right Strengthening	**Yes**	No	No
B8: Transitivity	Yes	**Yes**	No
B10: Left Weakening	No	No	**Yes**
B11: Rational Right Strengthening	No	No	**Yes**

Remark. Ordered abduction is logically weaker than non-reflexive abduction. However, the postulates of the former are not all weakened with respect to the latter (B1′ becomes stronger while B3 and B4 become weaker). One may wonder how this is possible. In fact, what matters is whether the transformation of postulates implies an increase or a decrease of the number of couples (E, H) such that $E \parallel\!\!\to H$. A postulate transformation is said to be ampliative (resp. restrictive) if it implies more (resp. less) couples. Any postulate states that "if antecedent then consequent" where antecedent and consequent contain one formula of type $E \parallel\!\!\to H$. It is easy to show the following:

- if consequent alone is weakened (resp. strengthened), the corresponding postulate is weakened (resp. strengthened) and ampliative (resp. restrictive);
- if antecedent alone is weakened (resp. strengthened), the corresponding postulate is strengthened (resp. weakened) and ampliative (resp. restrictive).

It can be checked that B1′ is submitted to a weakening of the antecedent, while B3 and B4 are submitted to a weakening of the consequent, hence all three are ampliative as it should be.

6.2. Comparison of Postulates

A first group of eight postulates are common to all abduction schemes.

A second group of three postulates differentiates non-reflexive and ordered abduction (and is common to ordered abduction and to non-transitive abduction). Reflexivity cannot be considered as a wishful postulate since nothing is gained if one abduces the fact that one wants to explain; however, it can be considered as some degenerated case which is not really harmful. Infra classicality and Right Strengthening correspond to an ideal deductive explanation scheme but are too demanding for common reasoning since they rule out most of the relevant abductions performed. A good illustration against Right Strengthening is given by Cialdea Mayer and Pirri (1996): the fact that some spoon of sugar has been added in my coffee is a good explanation of the fact that my coffee is sweet enough; but the fact that some spoon of sugar and some spoon of salt have been added is no more a good explanation of that sweetness. Both postulates are responsible for rejecting relevant hypotheses. Hence, their rejection is in favor of ordered abduction.

A third group of three postulates differentiates ordered and non-transitive abduction (and is common to non-reflexive and ordered abduction). Transitivity is an aimed property if one wants to proceed to abduction at higher and higher levels. Left Weakening and Rational Right Strengthening imply to abduce a lot of hypotheses which are not sufficiently sorted out. They are responsible for accepting abnormal hypotheses. Hence, their rejection is again in favor of ordered abduction.

Non-transitive abduction does not capture the intuitive properties of abduction very well. Non-reflexive abduction is generally unreachable for the reasons already detailed, but appears as a sort of ultimate aim. It can in fact be seen as a limit case of ordered abduction (as classical abduction is a limit case of non-reflexive abduction). Ordered abduction appears to obey to the best combination of postulates. In fact, the only remaining objection to ordered abduction is that it satisfies Reflexivity. This objection is not an argument for the other abduction schemes. It rather points out one limitation of the framework of belief revision: the notion of explanatory power is not embedded in the underlying preference relation on the set of possible worlds.

7. Conclusion

Two abduction schemes, non-reflexive and ordered abduction, were considered as serious candidates for representing the intuitive meaning of abduction. Ordered abduction was finally considered as the best definition of abduction. Non-reflexive abduction is considered as a sort of limit case which cannot be really reached due to the impossibility of clarifying all the provisos needed to reach a real classical deductive inference.

The paper is mainly oriented towards an epistemological and theoretical goal. It tries to make a link between abductive reasoning and other logical developments such as belief revision and non-monotonic inference. As such, further works could make the analysis deeper by extending the preceding definitions as well as the postulates. First, an infinite number of possible worlds would allow the modeler to deal with a larger set of propositions. Second, predicate logic instead of propositional logic would allow to deal with universal propositions, making easier the distinction between laws and facts. Third, the problem of the syntactical shape of abduction would also have to be considered. Lastly, probability calculus would favor the definition of the acceptability of a hypothesis and allow to build a bridge with diagnosis analysis often treated in a probabilistic framework.

Another direction of research would be to apply the ideas of the paper towards a more procedural and computational goal. This is precisely what abductive logical programming (ALP) intends to do. However, this very active field of research is not exempt of a more fundamental questioning. Quoting Denecker and Kakas (2001), "the definition of an abductive solution defines the formal correctness criterion for abductive reasoning, but does not address the question of how the ALP formalism should be interpreted.[. . .] For example, how is negation in ALP to be understood ? [. . .] Another open question is the relationship to classical logic." Hence, the two approaches should be thought as complementary appraisals of abductive reasoning but their precise links remain to be studied.

Appendix A: Representation Theorem for Non-Reflexive Abduction

Derived Propositions

B0′. If no hypothesis can be abduced from an event, then this event is empty.
It comes by recurrence from B1′ and B2.

(It is not a formal proposition hence cannot be incorporated in the set of postulates as one may wish in order to spare postulates B1′ and B2).

B1″. Weak Reflexivity: If (($E \ \|\!\prec H$) then ($H \ \|\!\prec H$)
B6 with $F = H$ gives $(E \cap H) \ \|\!\prec H$. By B3, if $E \ \|\!\prec H$ then $H \subseteq E$, hence $(E \cap H) = H$.

B8. Transitivity: If $[(E \ \|\!\prec F) \wedge (F \ \|\!\prec G)]$ then $(E \ \|\!\prec G)$
By B3 and B4.

B9. Left And: If $[(E \ \|\!\prec H) \wedge (F \ \|\!\prec H)]$ then $(E \cap F \ \|\!\prec H)$
From B3 and B6.

B46. Pointwise Left Strengthening: If $[(E \ \|\!\prec H) \wedge \neg \ (E \|\!\prec \mathrm{w})]$ then $(E \cap (-w) \|\!\prec H)$
If $[(E \ \|\!\prec H) \ \wedge \neg \ (E \ \|\!\prec \mathrm{w})]$ then $\neg(\mathrm{w} \subseteq H)$; otherwise, by B4 $[(E \|\!\prec H) \wedge (\mathrm{w} \subseteq H)]$ would give $(E \|\!\prec \mathrm{w})$. Hence, $[(E \ \|\!\prec H) \wedge \ H \subseteq (-\mathrm{w})]$ and then $(E \cap (-\mathrm{w}) \ \|\!\prec H)$ from B6.

B6′. If $[(E \ \|\!\prec H) \wedge (E \ \|\!\prec F) \wedge (F \ \subseteq \ H)]$ then $(H \ \|\!\prec F)$
By B6: $(E \cap H) \ \|\!\prec F$. By B3 $(H \subseteq \ E)$ hence $(E \cap H) = H$.

B26. If $[E \ \|\!\prec H) \wedge (G \subseteq E)]$ then $(H \cup G) \ \|\!\prec H$
From B1″ $H \ \|\!\prec H$ hence by B2 $E \cup H \ \|\!\prec H$. Now $[(E \cup H \ \|\!\prec H) \wedge (H \subseteq H \cup G)]$ and B6 give $[(E \cup H) \cap (H \cup G)] \ \|\!\prec H$. And $(E \cup H) \cap (H \cup G) = H \cup G$ if $(G \subseteq E)$.

B3″ If $E \|\!\prec H$ then $E \ \|\!\prec E \cap H$
Trivial because from B3 $E \cap H = H$.

B12. Weak Supra Classicality: If $E \ \|\!\prec E \ \wedge \ E \mid - H$, then $(E \ \|\!\prec H)$

Representation Theorem

THEOREM A.1. *Let * be a revision function satisfying AGM set of postulates* $\mathbf{A} = \{A1, A2, A3, A4, A5\}$, *then an inference relation* $\|\!\prec$ *defined according to* $(E \|\!\prec H) \equiv [(\emptyset \neq H \subseteq K * E)]$ *respects the set of postulates* $\mathbf{B}_{\mathrm{NR}} = \{B0, B1', B2, B3, B4, B5, B6, B7\}$ *and therefore is a non-reflexive abductive inference relation.*

Proof. (We will use equally $E \parallel\!\prec H$ or $H \subseteq K * E$ with $\emptyset \neq H$).

B0: trivial by definition.

B1′: Trivial because for every world w, $K|w = w$

B2: Let $E \parallel\!\prec F$ and $G \parallel\!\prec H$, i.e. $F \subseteq K * E$ and $H \subseteq K * G$. From A45: $K * (E \cup G) = K * E$ or $K * G$ or $(K * E \cup K * G)$. Hence, $F \subseteq K * (E \cup G)$ or $H \subseteq K * (E \cup G)$ hence $E \cup G \parallel\!\prec F$ or $E \cup G \parallel\!\prec H$.

B3: If $H \subseteq K * E$ then $H \subseteq E$ because $K * E \subseteq E$ by A2.

B4: Trivial.

B5: If $E \parallel\!\prec H$ and $E \parallel\!\prec G$ then $H \subseteq K * E$ and $G \subseteq K * E$. Then $G \cup H \subseteq K * E$ hence $E \parallel\!\prec G \cup H$.

B6: Assume $\emptyset \neq H \subseteq K * E$ and $H \subseteq F$. Then $H \subseteq K * E \cap F$. By A4: $K * E \cap F \subseteq K(^*E \cap F)$. Hence $H \subseteq K * (E \cap F)$.

B7: Assume $G \subseteq K * E, G \subseteq F, H \subseteq K * (E \cap F)$. By A5, $K * (E \cap F) \subseteq K * E \cap F$. Hence, $H \subseteq K * E$.

THEOREM A.2. *Let* $\parallel\!\prec$ *be a non-reflexive inference relation satisfying the set of postulates* $\mathbf{B}_{\mathrm{NR}} = \{\mathrm{B0, B1', B2, B3, B4, B5, B6, B7}\}$. *Then the operation* * *defined by* $K * E = \cup H, E \parallel\!\prec H$ *(union of all events abduced from* E*) where we set* $K = K * T$, *respects the set of postulates* $\mathbf{A} = \{\mathrm{A1, A2, A3, A4, A5}\}$ *and therefore it is a revision function. Moreover,* $(E \parallel\!\prec H) \equiv [(\emptyset \neq H \subseteq K * E)]$ *and* $K * E = \{w : E \parallel\!\prec w\}$.

Proof.

(a) We show first that $(E \parallel\!\prec H) \equiv [(\emptyset \neq H \subseteq K * E)]$.
If sense: If $\emptyset \neq H \subseteq K * E$ then $E \parallel\!\prec H$.
Let Abd(E) be the set of events abduced from E. By B5, Abd(E) is closed under union (W is finite). By B4, Abd(E) is closed under the sub-set operation. Let $\emptyset \neq H \subseteq K * E$. There exists a family $\{F_i\}$ of elements from Abd(E) such as $H \subseteq \cup F_i$. Now $\cup F_i \in$Abd(E) and since Abd(E) is closed under sub-set operation $H \in$ Abd(E) hence $E \parallel\!\prec H$.
Only if sense: If $E \parallel\!\prec H$ then $\emptyset \neq H \subseteq K * E$.
Trivial from the definition of $K * E$ and B0.
(b) Let us show now that $K * E = \{w, E \parallel\!\prec w\}$.
Let w be abduced from E. Then $\{w\} \subseteq K * E$ hence $w \in K * E$. Vice versa, let $w \in K * E$, hence there exist H such as $E \parallel\!\prec H$ and $\{w\} \subseteq H$ hence by B4 $E \parallel\!\prec \{w\}$.
(c) We can now prove that * is a belief revision function satisfying the postulates A1–A5.

A1: Assume $E \neq \emptyset$. If E is a single world then $E \parallel\!\prec E$ and $K * E = E \neq \emptyset$. If E contains more than a world, let $E = \cup w_i, i \in I$ with $I = \{1, 2, \ldots\}$. Now, $w_i \parallel\!\prec w_i$ for every i by B1′.

Then $w_1 \cup w_2 \parallel\!\prec w_1$ or $w_1 \cup w_2 \parallel\!\prec w_2$ by B2. Assume now that $\cup w_i \parallel\!\prec w_\alpha$ for $i, \alpha \in I' \subset I$. Let $j \in I\text{–}I'$. B2 gives: $(\cup w_i) \cup w_j \parallel\!\prec w_\alpha$ or $(\cup w_i) \cup w_j \parallel\!\prec w_j$. By recurrence, there exists some $\beta \in I$ such that $E \parallel\!\prec w_\beta$hence $K * E \neq \emptyset$. Moreover, this proves that in every case $K \neq \emptyset$ because $K = K * T$ and $T \neq \emptyset$.

A2: Trivial by B3.

A3: Assume $K \subseteq E$ then $K * T \subseteq E$. Le us show that $K * E = K = K * T$.

(a) Let $H \subseteq K * T$ then $T \parallel\!\prec H$ and $H \subseteq E$. By B6, $E \parallel\!\prec H$ then $H \subseteq K * E$. Then $K * T \subseteq K * E$.

(b) Let $H \subseteq K * E$. By A1, it exists $F \neq \emptyset$ such as $T \parallel\!\prec F$. Then from (a) $F \subseteq E$. Then $T \parallel\!\prec F$ and $F \subseteq E$ and $E \parallel\!\prec H$. By B7, $T \parallel\!\prec H$ hence $H \subseteq K * T$. Then $K * E \subseteq K * T$.

(*Remark*. This proof is unnecessary if we adopt the equivalent set of postulates $\{$A1, A2, A4, A5, $K * T = K\}$ for revision.)

A4: Let $H \subseteq (K * E) \cap F$. Then $E \parallel\!\prec H$ and $H \subseteq F$. Then by B6, $E \cap F \parallel\!\prec H$ hence $H \subseteq K * (E \cap F)$.

A5: Assume $(K * E) \cap F \neq \emptyset$. Then it exists G such as $E \parallel\!\prec G$ and $G \subseteq F$. By A1, $K * (E \cap F) \neq \emptyset$ because $(E \cap F) \neq \emptyset$ since $(K * E) \cap F \neq \emptyset$ and $K * E \subseteq E$. So let $H \subseteq K * (E \cap F)$ i.e. $E \cap F \parallel\!\prec H$. By B7, $E \parallel\!\prec H$ then $H \subseteq (K * E)$. But as $E \cap F \parallel\!\prec H$, $H \subseteq F$ by B3. Hence $H \subseteq (K * E) \cap F$.

Appendix B: Representation Theorem for Ordered Abduction

DERIVED PROPOSITIONS

B0. Non-contradiction: If $(E \parallel\!\approx H)$ then $(H \neq \emptyset)$
Trivial from B3′.

B14. Reflexive Weak Right Strengthening: If $[(G \subseteq E) \wedge (G \neq \emptyset)]$ then $[(E \parallel\!\approx G) \vee (E \cap (-G) \parallel\!\approx E)]$

From B4 with $E = H$ and B1. Moreover we can not have $E \parallel\!\approx G$ and $(E \cap (-G)) \parallel\!\approx E$; otherwise by B8 we would have $(E \cap (-G)) \parallel\!\approx G$ which is contradictory with B3′.

B2. Strong Left Or: If $[(E \parallel\!\approx F) \wedge (G \parallel\!\approx H)]$ then $[((E \cup G) \parallel\!\approx F) \vee ((E \cup G \parallel\!\approx H)]$

B14 with $E \subseteq E \cup G$ and $G \subseteq E \cup G$ proves that if neither $E \cup G \parallel\!\approx E$ nor $E \cup G \parallel\!\approx G$ then $[G \cap (-E) \parallel\!\approx E \cup G]$ and $[E \cap (-G) \parallel\!\approx E \cup G]$. Hence by B9 a contradiction with B3′. Then $[E \cup G \parallel\!\approx E]$ or $[E \cup G \parallel\!\approx G]$. Hence B2 through B8.

B2′. Left Or: If [$(E\|{\approx}F)\wedge(G\|{\approx}F)$] then $(E\cup G)\|{\approx}\, F$

Trivial from B2.

B26. If $[E\|{\approx}H)\wedge(G\subseteq E)]$ then $(H\cup G)\|{\approx}H$

From B1 $H\|{\approx}H$ hence by B2′ $E\cup H\|{\approx}H$. Now $[(E\cup H\|{\approx}\, H)\wedge(H\subseteq H\cup G)]$ and B6 give $[(E\cup H)\cap(H\cup G)]\ \|{\approx}H$. And $(E\cup H)\cap(H\cup G)=H\cup G$ if $(G\subseteq E)$.

B10. If $[(E\|{\approx}H)\wedge(G\subseteq E)\wedge\neg(E\|{\approx}G)]$ then $[E\cap(-G)]\ \|{\approx}H$

From B14 and B8

B7. Weak Cut: If $[(E\|{\approx}G)\wedge(G\subseteq F)\wedge(E\cap F\|{\approx}H)]$ then $(E\|{\approx}H)$

Assume that $[(E\|{\approx}G)\wedge(E\cap F\subseteq E)\wedge\neg(E\|{\approx}E\cap F)]$. Then by B10 $[E\cap(-E\cup -F)]\ \|{\approx}G$ i.e. $[E\cap(-F)]\|{\approx}G$. This is contradictory with $G\subseteq F$ by B3′. Hence $[(E\|{\approx}G)\wedge(G\subseteq F)]$ gives $E\|{\approx}E\cap F$. Then by B8, $[(E\|{\approx}G)\wedge(G\subseteq F)\wedge(E\cap F\|{\approx}H)]$ gives $(E\|{\approx}\, H)$.

B3″. If $E\|{\approx}H$ then $E\|{\approx}E\cap H$

Assume that $E\|{\approx}E\cap H$ is not the case. Then by B10 $[E\cap(-E\cap H)]\ \|{\approx}\, H$ i.e. $E\cap(-H)\|{\approx}H$. Hence a contradiction by B3′.

B67. If $[(E\|{\approx}H)\wedge(G\subseteq E)\wedge(H\cup G)\|{\approx}G]$ then $E\|{\approx}\, G$

By B4 $[(G\subseteq E)\wedge(H\cup G)\|{\approx}G]$ gives $((H\cup G)\cap E)\|{\approx}\, G$

By B5 $[(E\|{\approx}H)\wedge(H\subseteq(H\cup G))\wedge((H\cup G)\cap E)\|{\approx}\, G]$ gives $E\|{\approx}G$

Now, we show the equivalence between two sets of postulates, the second containing less postulates than the first.

THEOREM B.1. *The set of postulates* $\mathbf{BO}_{\mathrm{R}}=\{$B1, B3′, B4′, B5, B6, B8, B9$\}$ *and* $\mathrm{B}'_{\mathrm{R}}=\{$B3′, B14, B5, B6, B8, B9$\}$ *are equivalent.*

Proof. It suffices to prove that under the other postulates, B14 is equivalent to the conjunction of B1 and B4′.

We have already proved that B14 follows from the conjunction of B1 and B4′.

Conversely, assume B14. B1 follows immediately under B3′ if we set $E=G$. Let us show that B4′ follows equally. Assume that $(E\|{\approx}H)\wedge(\emptyset\neq G\subseteq H)$. Now by B15, from $(\emptyset\neq G\subseteq H)$, it follows that $[(H\|{\approx}G)\vee(H\cap(-G)\|{\approx}H)]$. If $(H\|{\approx}\, G)$ then by B8, $(E\|{\approx}G)$.

So, to complete the proof, it suffices to show that: if $(E\|{\approx}H)\wedge(\emptyset\neq G\subseteq H)\wedge(H\cap(-G)\|{\approx}H)$ then $E\cap(-G)]\|{\approx}E$. In this case, we do not have $H\|{\approx}G$ (see the proof above).

(a) If $G\subseteq(-E)$, then $E\cap(-G)=E$. Hence by B1, $E\cap(-G)\ )\|{\approx}E$
(b) If $G\subseteq E$, then from B14 it follows that $[(E\|{\approx}G)\vee(E\cap(-G)\|{\approx}E)]$. Let us show that we have not $E\|{\approx}G$. If we assume the opposite, then we have $(E\|{\approx}\, G)\wedge(G\subseteq H)\wedge(G\subseteq E)$. Then $G\subseteq E\cap H$. Then from B14, it follows

that either $E \cap H \|{\approx} G$ or $G \cap (-(E \cap H)) \|{\approx} (E \cap H)$. The latter case is impossible since $G \cap (-(E \cap H)) = \emptyset$. So $E \cap H \|{\approx} G$. Now by B14, from $E \cap H \subseteq H$ it follows that either $H \|{\approx} E \cap H$ or $(H \cap (-(E \cap H)) \|{\approx} H$. The latter case is impossible because $H \cap (-(E \cap H)) = H \cap (-(E))$; so we would have $H \cap (-(E)) \|{\approx} H$ which is contradictory to $E \|{\approx} H$ under B9. So $H \|{\approx} E \cap H$. Then by B8, $H \|{\approx} G$ in contradiction with the hypothesis. So we have not $E \|{\approx} G$. Hence, we have $E \cap (-G) \|{\approx} E$.

(c) In general $G = G_1 \cup G_2$ with $G_1 \subseteq E$ and $G_2 \subseteq (-E)$. So $E \cap (-G) = E \cap (-G_1)$. So it suffices to prove that the conditions respected by G are respected by G_1 and to use the proof b). The only point to show is that if $(G_1 \cup G_2 \subseteq H)$ and if we have not $(H \|{\approx} G_1 \cup G_2)$ then we do not have $H \|{\approx} G_1$. Or, what is equivalent, if $(G_1 \cup G_2 \subseteq H)$ and $H \|{\approx} G_1$ then $H \|{\approx} G_1 \cup G_2$. Now by B1, $H \cap (G_1 \cup G_2) = (G_1 \cup G_2) \|{\approx} G_1 \cup G_2$. Then from B7 (we can use it as it follows from other postulates than B4′): if $[(H \|{\approx} G_1) \wedge (G_1 \subseteq G_1 \cup G_2) \wedge (H \cap (G_1 \cup G_2) \|{\approx} (G_1 \cup G_2)]$ then $H) \|{\approx} (G_1 \cup G_2)$.

REPRESENTATION THEOREM

THEOREM B.2. *Let * be a revision function satisfying AGM set of postulates* $\mathbf{A} = \{A1, A2, A3, A4, A5\}$, *then an inference relation* $\|{\approx}$ *defined according to* $(E \|{\approx} H) \equiv [(\emptyset \neq K * H \subseteq K * E)]$ *respects the set of postulates* $\mathbf{Bo}_{\mathrm{R}} = \{B1, B3', B4', B5, B6, B8, B9\}$ *and therefore is a reflexive abductive inference relation.*

Proof.

B1: Trivial

B3′: Let $(E \|{\approx} H)$ then $\emptyset \neq K * H \subseteq K * E$. Then by A2 $K * H \subseteq H$ and $K * E \subseteq E$. Hence $K * H \subseteq E \cap H \neq \emptyset$.

B4′: Let $(E \|{\approx} H)$ and $(G \subseteq H)$.

Assume first that $G \cap K * H \neq \emptyset$. As $G = (G \cap H)$, $K * G = K * (G \cap H)$. Hence by A4 and A5 $K * G = G \cap K * H \subseteq K * H$. Now $K * H \subseteq K * E$ hence $K * G \subseteq K * E$ i.e. $E \|{\approx} G$.

Assume now that $G \cap K * H = \emptyset$. Now $K * H \subseteq K * E \subseteq E$ and $K * H \subseteq H$ by A2. So $K * E \cap H \neq \emptyset$. Hence $K * H = K * H \cap E = K * (E \cap H) = K * E \cap H$. Then $G \cap K * H = \emptyset$ gives $G \cap K * E \cap H = \emptyset$ then $G \cap K * E = \emptyset$ i.e. $K * E \subseteq -G$. Then $K * E \cap (-G) = K * E \neq \emptyset$. Then by A4 and A5, $K * (E \cap (-G)) = K * E \cap (-G) = K * E$. Hence $E \cap (-G) \|{\prec} E$.

B5: Let $(E \|{\approx} F) \wedge (E \|{\approx} H)$ then $[(K * F \subseteq K * E) \wedge (K * H \subseteq K * E)]$. A2, A4 and A5 gives A45 (Right Distributivity) then $K * (F \cup H)$ is equal to either $K * F$ or $K * H$ or $K * F \cup K * H$. Hence $K * (F \cup H) \subseteq K * E$. Hence $[E \|{\approx} (F \cup H)]$.

B6: Let $[(E\|{\approx}H)\wedge (H \subseteq F)]$ then $[(K * H \subseteq K * E)\wedge (H \subseteq F)]$. By A2, $K * H \subseteq H$. Then $K * H \subseteq F$. By A4 $K * H \subseteq K * E \cap F \subseteq K * (E \cap F)$. Hence $(E \cap F\|{\approx} H)$.

B8: Let $[(E\|{\approx}F)\wedge (F\|{\approx}G)]$ then $K * F \subseteq K * E$ and $K * G \subseteq K * F$ then $K * G \subseteq K * E$ hence $(E\|{\approx}G)$.

B9: Let $[(E\|{\approx}H)\wedge (F\|{\approx}H)]$ then $K * H \subseteq K * E$ and $K * H \subseteq K * F$. By A2, $K * F \subseteq F$ then $K * H \subseteq K * E \cap F$. Hence by A4, $K * H \subseteq K * (E \cap F)$ then $[(E \cap F)\|{\approx} H]$.

THEOREM B.3. *Let* $\|{\approx}$ *be a reflexive inference relation satisfying the set of postulates* $\mathbf{Bo}_{\mathrm{R}} = \{B1, B3', B4', B5, B6, B8, B9\}$. *Then the operation* $*$ *defined by* $K * E = \cap H, H\|{\approx} E$ *(intersection of all events from which E can be abduced) and where we set* $K = K * T$*, respects the set of postulates* $\mathbf{A} = \{A1, A2, A3, A4, A5\}$ *and therefore is a revision function. Moreover,* $(E\|{\approx}H) \equiv [(\emptyset \neq K * H \subseteq K * E)]$ *and* $K * E = \{w, E\|{\approx}w\}$.

Proof.

(a) We show first that $(E\|{\approx}H)) \equiv (\emptyset \neq K * H \subseteq K * E)$
If sense: if $(\emptyset \neq K * H \subseteq K * E)$ then $(E\|{\approx}H)$
Let $(K * H \subseteq K * E)$ hence if $(F\|{\approx}E)$ then $(K * H \subseteq F)$. Then $K * H \subseteq E$ because $E\|{\approx}E$. Then by B15, $E\|{\approx}K * H$ or $E \cap (-K * H)\|{\approx}E$. But if $E \cap (-K * H)\|{\approx} E$ then $K * H \subseteq E \cap (-K * H)$ which is impossible. Then $E\|{\approx}K * H$. Now $K * H\|{\approx}H$ by B9 so $E\|{\approx}H$ by B8.
Only if sense: If $(E\|{\approx}H)$ then $(\emptyset \neq K * H \subseteq K * E)$
$K * H = \cap G/G\|{\approx}H$ and $K * E = \cap F/F\|{\approx}E$. Assume $(E\|{\approx}H)$. By B8, if $(F\|{\approx}E)$ then $(F\|{\approx}H)$. Hence $\{F/F\|{\approx}E\} \subseteq \{G/G\|{\approx}H\}$. Then $[\cap G/G\|{\approx}H] \subseteq [\cap F/F\|{\approx}E]$ hence $(K * H \subseteq K * E)$.
Now, by B9, $[\cap G/G\|{\approx}H]\ \|{\approx}H$ then by B3′, $K * H \cap H \neq \emptyset$

(b) Let us show now that $K * E = \{\mathrm{w}; E\|{\approx}\mathrm{w}\}$.
Let $\mathrm{w}/E\|{\approx}\mathrm{w}$ then $\mathrm{w} \subseteq \{\cap H/H\|{\approx}E\}$. Indeed, $\mathrm{w} \subseteq \{\cap H, H\|{\approx} E\}$ is equivalent to (if $H\|{\approx}E$ then $\mathrm{w} \subseteq H$). Now $H\|{\approx}E$ and $E\|{\approx}\mathrm{w}$ imply $H\|{\approx}\mathrm{w}$ by B8. Then $H \cap \mathrm{w} \neq \emptyset$ by B3′ hence $\mathrm{w} \subseteq H$.
Conversely, let w such as if $H\|{\approx}E$ then $\mathrm{w} \subseteq H$. Then $\mathrm{w} \subseteq E$ because $E\|{\approx}E$. Assume $E\|{\approx}\mathrm{w}$ is not the case. Then by B10, from $[(E\|{\approx}E)\wedge(\mathrm{w} \subseteq E) \wedge \neg(E\|{\approx}\mathrm{w})]$, one obtains $(E \cap (-w)\|{\approx}E)$. Now $[\mathrm{w} \subseteq (E \cap (-w))]$ is not the case and this is in contradiction with [if $H\|{\approx}E$ then $\mathrm{w} \subseteq H$].

(c) We can now prove that the postulates are satisfied. Since by definition $K * T = K$, it is enough to show that {A1, A2, A4, A5} is respected, by using the set of postulates equivalent to **A**.

A1. By B1, $\emptyset \neq E\|{\approx}E$. So there exists at least one H such as $H\|{\approx} E$. By B9, $[\cap H/H\|{\approx}E]\ \|{\approx}E$ i.e. $K * E\|{\approx}E$. By B3′, $K * E \cap E \neq \emptyset$. The same reasoning with $E = T$ shows that $K = K * T$ is never empty.

A2. By B1 Ø $\neq$E$\|\!\approx$E then $[\cap H/H\|\!\approx$E$] \subseteq$ E.

A4. We prove first the following *corollary*: If $G \subseteq K * E$ then $K * G = G$

It is enough to show that $G \subseteq K*G$ (the other direction comes from B2). Let us show that if $G\cap(-K*G) \neq \emptyset$ then $G\|\!\approx[G\cap(-K*G)]$ which is contradictory as it means $K*[G\cap(-K*G)] \subseteq K*G$ when by A2 $K*[G\cap(-K*G)] \subseteq G\cap(-K*G)$.

By B15, $\emptyset \neq [G \cap (-K * G)] \subseteq E$ implies either $E\|\!\approx[G \cap (-K * G)]$ or $E \cap [-(G \cap (-K * G))]\|\!\approx E$. In this latter case, $[E\backslash(G \cap (-K * G))]\|\!\approx E$. Then, $K*E \subseteq K*[E-(G\cap(-K*G))]$ hence by A2, $K*E \subseteq [E-(G\cap(-K*G))]$ which is contradictory because $[G \cap (-K * G)] \subseteq K * E$. Hence $E\|\!\approx[G \cap (-K * G)]$.

By B6, $E\|\!\approx[G \cap (-K * G)]$ and $[G \cap (-K * G)] \subseteq G$ imply $(E \cap G)\|\!\approx[G \cap (-K * G)]$ then $G\|\!\approx[G \cap (-K * G)]$. As we have shown that it is contradictory, then $G \cap (-K * G) = \emptyset$. □

We prove now A4.

B4 shows that If $[(E\|\!\approx H)\wedge\ (H \subseteq F)]$ then $(E \cap F\|\!\approx H)$, hence if $[(K * H \subseteq K * E) \wedge (H \subseteq F)]$ then $K * H \subseteq K * (E \cap F]$. Let $G \subseteq (K * E) \cap F$. We have $K * G \subseteq G \subseteq K * E$ and $G \subseteq F$. Then $K * G \subseteq K * (E \cap F)$. Now $K * G = G$ by the corollary. This shows that $(K * E) \cap F \subseteq K * (E \cap F)$.

A5. Assume that $((K * E) \cap F \neq \emptyset)$ then $(K * (E \cap F) \subseteq (K * E) \cap F)$.

By B15, $[(E \cap F) \subseteq E) \wedge ((E \cap F) \neq \emptyset)]$ implies $E\|\!\approx(E \cap F)$ or $(E \cap (-F))\|\!\approx E$. Then $(K * (E \cap F) \subseteq K * E$ or $K * E \subseteq K * [E \cap (-F)]$. But by A2, $K * [E \cap (-F)] \subseteq [E \cap (-F)]$ which is contradictory with $((K * E) \cap F \neq \emptyset)$. Then $(K * (E \cap F) \subseteq K * E$. And by A2, $K * (E \cap F) \subseteq E \cap F$.

Acknowledgements

We thank Paul Bourgine for its contribution to the global taxonomy, David Makinson and Jerome Lang for several constructive suggestions, the participants in seminars in Paris (cognitive economics) and Torino (LOFT 5 Conference) and two anonymous referees for helpful remarks (especially concerning the problem in Section 4.1).

References

Alchourron, C.E., Gärdenfors, P., and Makinson, D., 1985, "On the logic of theory change: Partial meet contraction and revision functions," *Journal of Symbolic Logic* **50**, 510–530.

Boutilier, C. and Becher, V., 1995, "Abduction as belief revision," *Artificial Intelligence* **77**(1), 43–94.

Cialdea Mayer, M. and Pirri, F., 1996, "Abduction is not deduction-in-reverse," *Journal of the IGPI* **4**(1), 1–14.

Darwiche, A. and Pearl, J., 1997, "On the logic of iterared belief revision," *Artificial Intelligence* **89**, 1–29.

Denecker, M. and Kakas, A., 2001, "Abduction in logic programming, in *Computational Logic: Logic Progamming and Beyond (in honour of Robert A. Kowalski)*, Lectures Notes in Artificial Intelligence, Springer Verlag.

Flach, P., 1996, "Rationality postulates for induction," in *TARK VI: Sixth Conference on Theoretical Aspects of Rationality and Knowledge*, Morgan Kaufmann.

Flach, P. and Kakas, A., 2000, *Abduction and Induction*, Kluwer Academic Publishers, Dordrecht.

Harman, G.H., 1978, "The inference to the best explanation," *Philosophical Review* **71**, 88–95.

Hempel, C.G., 1965, *Aspects of Scientific Explanation and Other Essays in the Philosophy of Science*, The Free Press.

Hempel, C.G., 1988, "A problem concerning the inferential function of scientific theories," in *The Limitations of Deductivism*, A.Grünbaum and W.C.Salmon, eds., University of California Press.

Kraus, S., Lehmann, D., and Magidor, M., 1990, "Non monotonic reasoning, preferential models and cumulative logics," *Artificial Intelligence* **44**, 167–208.

Lehmann, D. and Magidor, M., 1992, "What does a conditional base entail?" *Artificial Intelligence* **55**, 1–60.

Levi, I., 1979, "Abduction and demands for information," pp. 405–29 in *The Logic and Epistemology of Scientific Change*, I. Niinuloto and R.Tuomela, eds., North Holland for Societas Philosophica Fennica, Amsterdam. Reprinted in I. Levi, 1984, *Decisions and Revisions*, Cambridge University Press.

Lipton, P., 1991, *Inference to the best Explanation*, Routledge.

Lobo, J. and Uzcategui, C., 1997, "Abductive consequence relations," *Artificial Intelligence*, **89**, 149–171.

Makinson, D., 1993, "Five faces of minimality," *Studia Logica* **52**, 339–379.

Makinson, D. and Gardenfors, P., 1991, "Relations between the logic of theory change and non-monotonic logic," pp. 185–205 in *The Logic of Theory Change*, A. Fuhrman and M. Morreau, eds., Springer.

Peirce, C.S., 1931–1958, in C. Hartshorne and P. Weiss, eds., *Collected Papers of Charles Sanders Peirce*, Harvard University Press.

Pino-Perez, R. and Uzcategui, C., 1999, "Jumping to explanations versus jumping to conclusions," *Artificial Intelligence* **111**, 131–169.

Poole, D., 1988, "A logical framework for default reasoning," *Artificial Intelligence* **36**, 27–47.

Poole, D., 1989, "Explanation and prediction: an architecture for default and abductive reasoning," *Computational Intelligence* **6**, 97–110.

Popper, K.R., 1959, *The Logic of Scientific Discovery*, Hutchinson & Co.

Rescher, N., 1978, *Peirce's Philosophy of Science: Critical Studies in his Theory of Induction and Scientific Method*," University of Notre Dame Press.

Stalnaker, R., 1968, "A theory of conditionals," in N. Rescher, ed., *Philosophical Quarterly Monograph Series*, Vol. 2, Basil Blackwell.

Thagard, P., 1978, "The best explanation: Criteria for theory choice," *Journal of Philosophy* **75**, 76–92.

van Fraassen, B.C., 1980, *The Scientific Image*, Oxford University Press, Oxford.

Zwirn, D., Zwirn, H., 1996, "Metaconfirmation," *Theory and Decision* **3**, 195–228.

Queries

Q1. Au: Please provide Figure caption.

Q2. Author: Please provide caption of Table I

Q3. Au: Edit ok?

Q4. Au: Please provide caption of Table II.